L21: Dealing with nuisance parameters

1. Go over problems in Exam 3.

2. Sufficient statistic and test

   (1) Definition and factorization theorem
   $X = (X_1, ..., X_n)$ is a random sample from a distribution with joint pdf or pmf $f(x;\,\eta)$ where η is unknown parameter (vector). Let $T(X)$ be a statistic (vector) and $f_T(\cdot)$, $f_{X|T}(\cdot)$ be pdf/pmf of T and conditional pdf or pmf of X given T.

   $$\begin{array}{rcl} T \text{ is a sufficient statistic for } \eta & \overset{def}{\Longleftrightarrow} & f_{X|T}(x) \text{ is free of } \eta \\ & \overset{*}{\Longleftrightarrow} & f(x;\,\eta) = g_1(x)\cdot g_2(t(x);\,\eta) \end{array}$$

   Proof. $\overset{*}{\Rightarrow}$**:** $f(x;\,\eta) = f_{X|T}(x)\cdot f_T(t;\eta) = g_1(x)\cdot g_2(t(x);\,\eta)$.

   $\overset{*}{\Leftarrow}$**:** Let $y = \begin{pmatrix} t(x) \\ y_*(x) \end{pmatrix} \Longleftrightarrow X = \begin{pmatrix} x_I(t, y_*) \\ x_{II}(t, y_*) \end{pmatrix}$ be 1-1 mapping with $J(t, y_*) = \left|\frac{\partial x}{\partial y^T}\right|$.

   $$\begin{array}{rcl} \text{Then } f_T(\cdot) & = & \iint f_Y(t, y_*)d\mu(y_*) = \iint f(x;\,\eta)J(t, y_*)d\mu(y_*) \\ & = & \iint g_1(\cdot)g_2(\cdot)J(\cdot)d\mu(y_*) = g_2(\cdot)\iint g_1(\cdot)J(\cdot))dy_2, .., d\mu(y_*) \\ & = & g_2(t;\,\eta)g_3(t) \end{array}$$

   So $f_{X|T}(\cdot)f_T(\cdot) = f(\cdot) \Longrightarrow f_{X|T}(\cdot)g_2g_3 = g_1g_2 \Longrightarrow f_{X|T}(\cdot) = g_1/g_3$ is free of η.

   (2) Critical function and sufficient statistic
   T is a sufficient statistic for η. For critical function $0 \le \psi(X) \le 1$ with $\beta_\psi(\eta) = E_\eta[\psi(X)]$ there exists $0 \le \psi_*(T) \le 1$ such that $\beta_{\psi_*}(\eta) \equiv \beta_\psi(\eta)$.

   Proof. Let $\psi_*(T) = E_\eta[\psi(X)|T]$.
   T is sufficient for $\eta \Longrightarrow \psi_*(T)$ contains no η.
   $0 \le \psi(X) \le 1 \Longrightarrow 0 \le E[\psi(X)|T] \le 1 \Longrightarrow 0 \le \psi_*(T) \le 1$.
   $\beta_{\psi_*}(\eta) = E_\eta[\psi_*(T)] = E_\eta\{E_\eta[\psi(X)|T]\} = E_\eta[\psi(X)] = \beta_\psi(\eta)$.

   (3) The essence of constructing a test scheme
   When constructing a test scheme, there is a sample X with distribution depending on parameter η. The goal is to construct $0 \le \phi(X) \le 1$ such that $E_\eta[\phi(X)]$ satisfies a set of restrictions.
   But with sufficient statistic T, by (2), this task becomes to construct $0 \le \phi(T) \le 1$ such that $E_\eta[\phi(T)]$ satisfies a set of restrictions.

   Ex1: The pdf or pmf of sample X $f(x;\,\eta) = 1\cdot f(x;\,\eta)$, By factorization theorem, sample X is sufficient for η. Thus the original version of constructing a test fits the new version.

   Comments: Suppose T_1 and T_2 are two statistics, we write $T_1 \preceq T_2$ if T_1 is a function of T_2. Then relation $\preceq$ is a reflexive and transitive, and hence is called an order.
   If $T_1 \preceq T_2$ and T_1 is sufficient for η, then T_2 is sufficient for η. But if T_2 is sufficient, T_1 may or may not be sufficient.
   If T is sufficient for η and $T \preceq S$ for all sufficient statistic S, then T is the minimal sufficient statistic. Minimal sufficient statistic is the simplest sufficient statistic.

3. Nuisance parameter and conditional test

(1) Settings

X is a sample from a population with parameter vector $\eta = \begin{pmatrix} \theta \\ \tau \end{pmatrix}$. $\begin{pmatrix} T(X) \\ S(X) \end{pmatrix}$ is a statistic vector. Suppose X has joint pdf or pmf

$$f(x;\, \theta,\, \tau) = g_1(x) g_2(t(x);\, \theta) g_3(s(x); \tau).$$

Then by factorization theorem, $\begin{pmatrix} T \\ S \end{pmatrix}$ is sufficient for $\eta = \begin{pmatrix} \theta \\ \tau \end{pmatrix}$; T is sufficient for θ; and S is sufficient for τ.

(2) Nuisance parameter when testing on θ

To construct a test on a hypothesis about θ, because T is sufficient for θ, we need to create $0 \le \phi(T) \le 1$ with a set of desired properties on $E[\phi(T)]$.

But the distribution of T depends on both θ and τ; and $E[\phi(T)]$ is a function of both θ and τ. While θ is the parameter of interest, τ becomes a nuisance parameter. We need to "hide" or "remove" this nuisance parameter.

(3) Conditional tests

Because S is sufficient for τ, the conditional distribution of T given S is free of τ, and $E[\phi(T)|S]$ is free of τ. Thus the nuisance parameter τ is successfully removed by using the conditional distribution given S.

Thus a test with critical function $0 \le \phi(T) \le 1$ satisfying a set of restrictions on $E_\theta[\phi(T)|S]$ is called a conditional test.

L22 Conditional tests

1. Conditional test on simple H_0 and simple H_a

   (1) Simple H_0 and simple H_a
   For $H_0 : \theta = \theta_0$ versus $H_a : \theta = \theta_1$ with sample pdf(pmf) $f(x;\, \theta)$,
   let $\Lambda = \frac{f(X;\theta_1)}{f(X;\theta_0)} = \Lambda(X)$. Then

$$\phi(X) = \begin{cases} 1 & \Lambda > c \\ r & \Lambda = c \\ 0 & \Lambda < c \end{cases} \quad \text{with } E_{\theta_0}[\phi(X)] = \alpha$$

   is α-level MP test.

   (2) Dealing with nuisance parameter
   If sample pdf (pmf) is $f(X;\, \theta,\, \tau)$, then $\Lambda = \frac{f(X;\theta_1,\tau)}{f(X;\theta_0,\tau)} = \Lambda(X;\, \tau)$ is no longer a statistic.
   In such a case one has to assume that there is a sufficient statistic S for τ.
   Let $\Lambda = \frac{f_{X|S}(X;\theta_1)}{f_{X|S}(X;\theta_0)} = \Lambda(X,\, S)$. Then

$$\phi(X) = \begin{cases} 1 & \Lambda > c \\ r & \Lambda = c \\ 0 & \Lambda < c \end{cases} \quad \text{with } E_{\theta_0}[\phi(X)|S] = \alpha$$

   is α-level conditional MP test on S.

   (3) Making use of sufficient statistic for θ
   Suppose that T is sufficient for θ and S is sufficient for τ.
   Let $\Lambda = \frac{f_{T|S}(T;\theta_1)}{f_{T|S}(T;\theta_0)} = \Lambda(T,\, S)$. Then

$$\phi(T) = \begin{cases} 1 & \Lambda > c \\ r & \Lambda = c \\ 0 & \Lambda < c \end{cases} \quad \text{with } E_{\theta_0}[\phi(T)|S] = \alpha$$

   is α-level conditional MP test.

   Comment: Because $\Lambda = \Lambda(T,\, S)$, in essence the critical function $\phi = \phi(T,\, S)$.

   Ex1: If $f(X;\, \theta,\, \tau) = g_1(\theta,\, \tau)g_2(X)g_3(\theta,\, T(X))g_4(\tau,\, S(X))$, then by factorization theorem, $T(X)$ is sufficient for θ and $S(X)$ is sufficient for τ.
   Let $y = (t(x),\, s(x),\, y_*(x)) = y(x) \Longleftrightarrow x = x(t,\, s,\, y_*) = x(y)$ be 1-1 mapping with $J(t,\, s,\, y_*) = \left|\frac{\partial x}{\partial y^T}\right|$. Then the pdf (pmf) for $Y = (T,\, S, Y_*)$ is

$$f_{(T,S,Y_*)}(t,\, s,\, y_*;\, \theta,\, \tau) = g_1(\theta,\, \tau)g_2(t,\, s,\, y_*)g_3(\theta,\, t)g_4(\tau,\, s)J(t,\, s,\, y_*).$$

   By the integration or summation over y_*, the pdf (pmf) for $(T,\, S)$ is

$$f_{(T,S)}(t,\, s;\, \theta,\, \tau) = g_1(\theta,\, \tau)g_3(\theta,\, t)g_4(\tau,\, s)h(t,\, s).$$

   By integration or summation over t, the pdf (pmf) for S is

$$f_S(t,\, \theta,\, \tau) = g_1(\theta,\, \tau)g_4(\tau,\, s)h_S(\theta,\, s).$$

Thus the conditional pdf (pmf) of T given S,

$$f_{T|S}(\cdot) = \frac{f_{(T,\,S)}(\cdot)}{f_S(\cdot)} = g_3(\theta,\, t)h(t,\, s)/h_s(\theta,\, s)$$

is a function of t, depends on s and θ, but is free of τ.

2. Conditional test with upper-sided H_a

   (1) Upper-sided H_a
   For testing on $H_0 : \theta \le \theta_0$ versus $H_a : \theta > \theta_0$, the monotone likelihood ratio in $T(X)$, i.e.,

   $$\theta_1 < \theta_2 \Longrightarrow \frac{f(x;\, \theta_2)}{f(x;\, \theta_1)} \text{ is an increasing function of } T(X)$$

   is required. Under this assumption

   $$\phi(T) = \begin{cases} 1 & T > c \\ r & T = c \\ 0 & T < c \end{cases} \quad \text{with } E_{\theta_0}[\phi(T)] = \alpha$$

   is α-level UMP test.

   (2) The problem with nuisance parameter
   If $f(x;\, \theta,\, \tau) = g_1(\theta,\, \tau)g_2(x)e^{r(\theta)t(x)}g_4(\tau,\, s(X))$ where $r(\theta)$ is an increasing function, then

   $$\theta_1 < \theta_2 \Longrightarrow \frac{f(x;\, \theta_2,\, \tau)}{f(x;\, \theta_1,\, \tau)} = \frac{g_1(\theta_2,\, \tau)}{g_1(\theta_1,\, \tau)}e^{[r(\theta_2)-r(\theta_2)]t(x)} \text{ is an increasing function of } T(X)$$

   But $E_{\theta_0}[\phi(T)]$ dependes on τ. Hence $\phi(T)$ in (1) can not be established.

   (3) Conditional test
   By factorization, $S(X)$ is sufficient for τ. Hence the distribution of T given S is free of τ. Thus $\phi(T)$ with $E_{\theta_0}[\phi(T)|S] = \alpha$ is a conditional UMP test.

   Ex2: With $g_3(\theta,\, t) = e^{r(\theta)t}$ where $r(\theta)$ is an increasing function, by Ex1,

   $$f_{(T,\,S)}(t,\, s;\, \theta,\, \tau) = g_1(\theta,\, \tau)e^{r(\theta)t}g_4(\tau,\, s)h(t,\, s).$$

   So $f_T(t;\, \theta,\, \tau) = g_1(\theta,\, \tau)e^{r(\theta)t}h_t(\tau,\, t)$ depends on nuisance τ.

   Ex3: By Ex1, $f_{T|S}(t;\, \theta|s) = e^{r(\theta)t}h(t,\, s)/h_s(\theta,\, s)$. Thus

   $$\theta_1 < \theta_2 \Longrightarrow \frac{f_{T|S=s}(t;\, \theta_2)}{f_{T|S=s}(t;\, \theta_1)} = \frac{h_s(\theta_1,\, s)}{h_s(\theta_2,\, s)}\, e^{[r(\theta_2)-r(\theta_1)]t} \text{ is an increasing functin of } T.$$